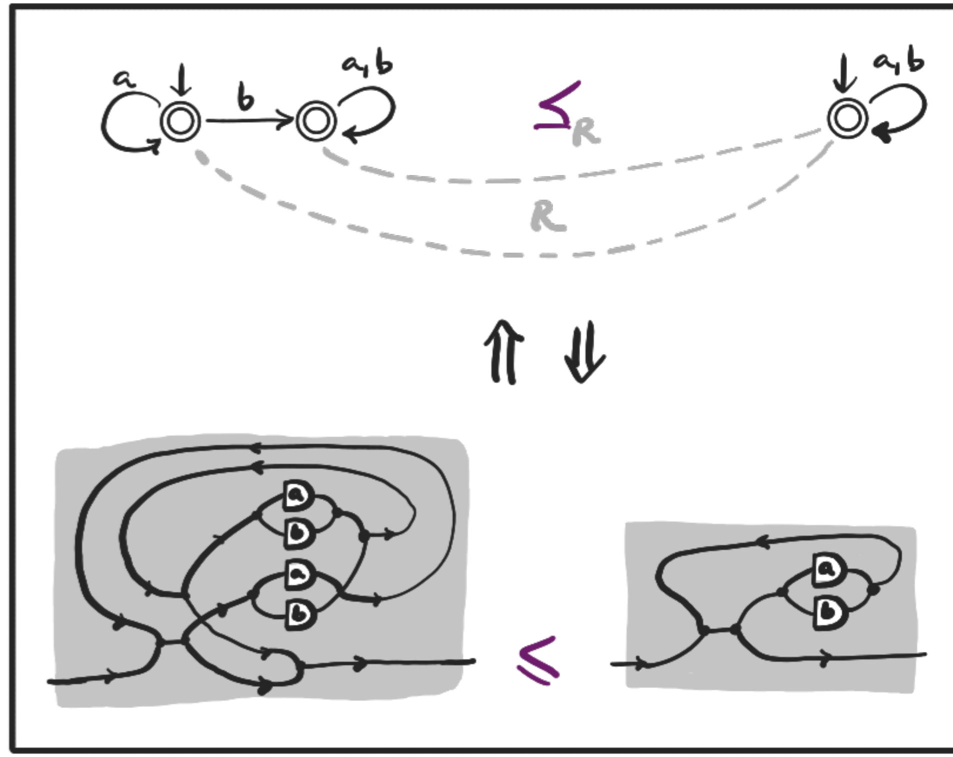
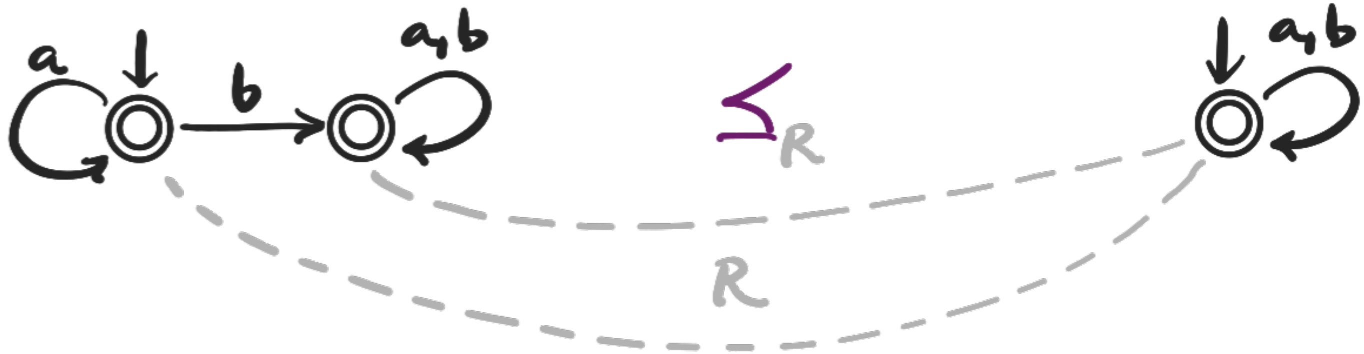
A COMPLETE DIAGRAMMATIC CALCULUS for AUTOMATA SIMULATION



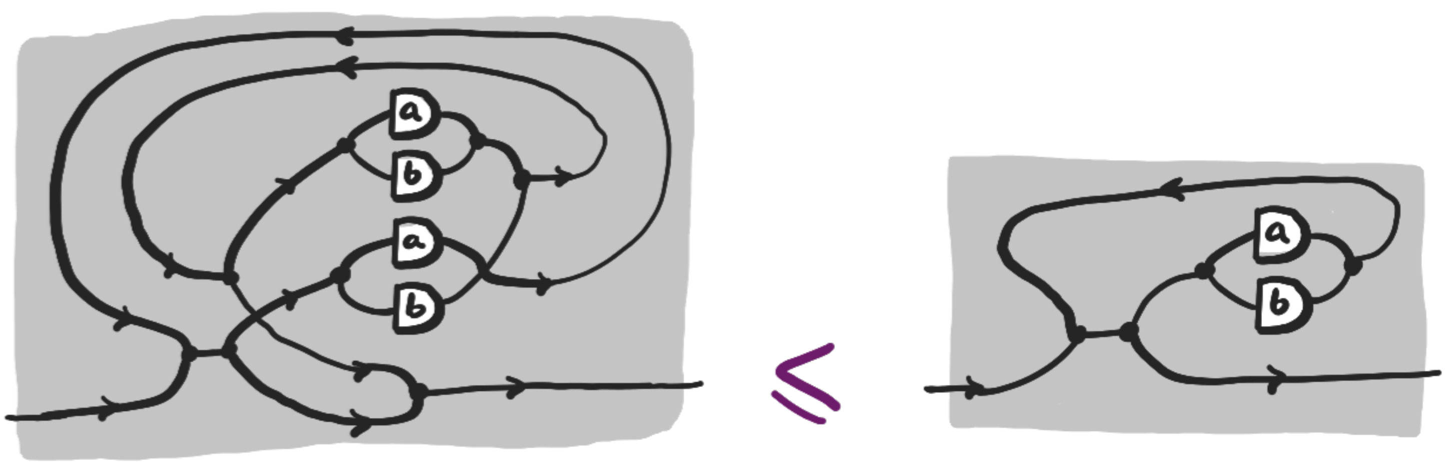
Thibaut Antoine[1], Robin Piedeleu[2], Alexandra Silva[2], Fabio Zanasi[2]

[1] ENS Rennes [2] UCL

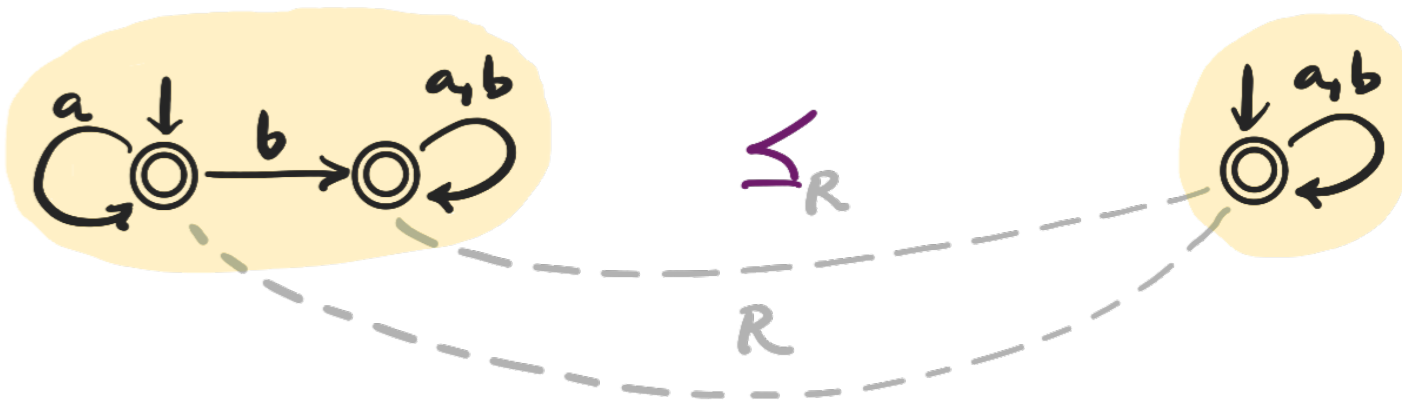
ONE-SLIDE SYNOPSIS



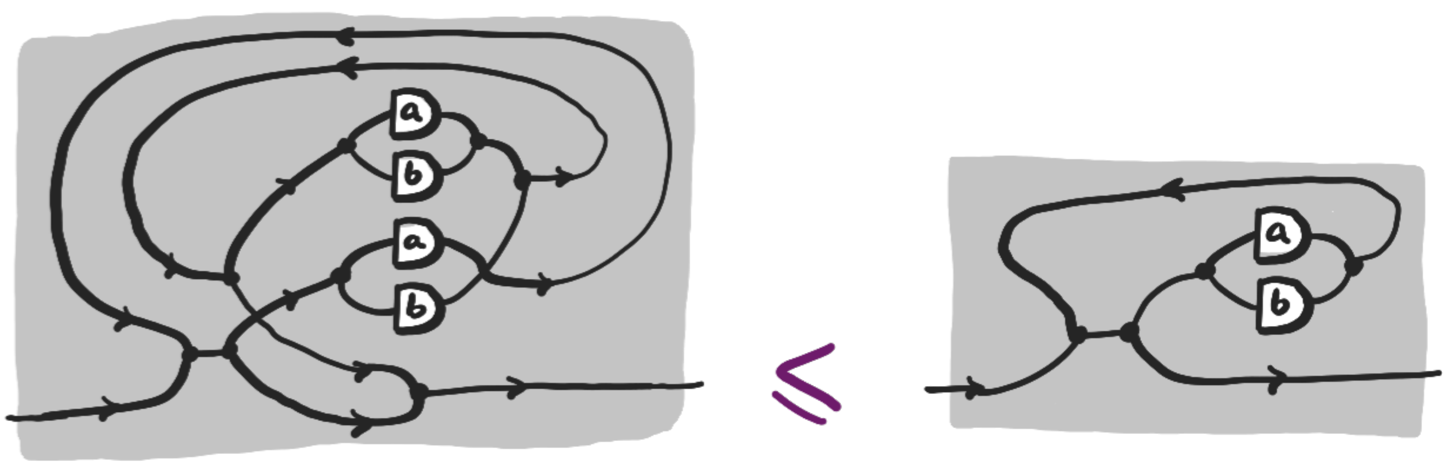
SOUNDNESS $\Uparrow$ $\Downarrow$ COMPLETENESS



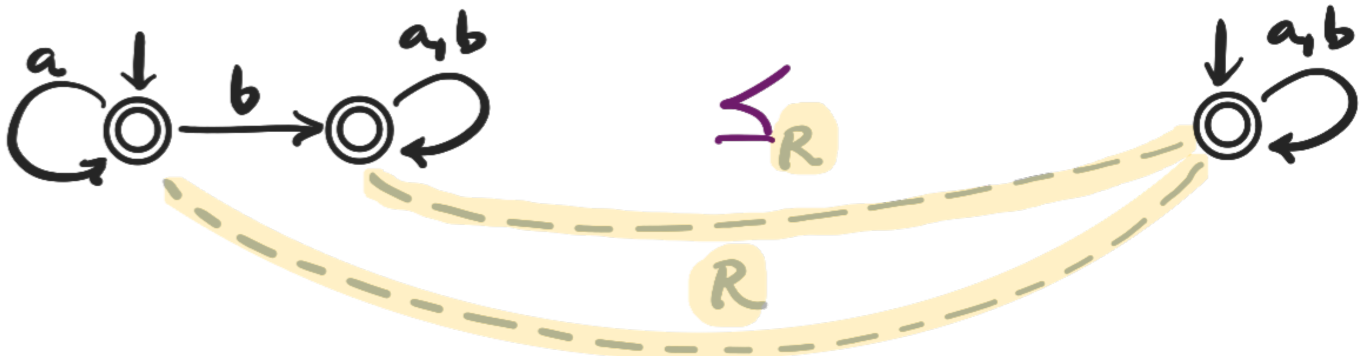
ONE-SLIDE SYNOPSIS



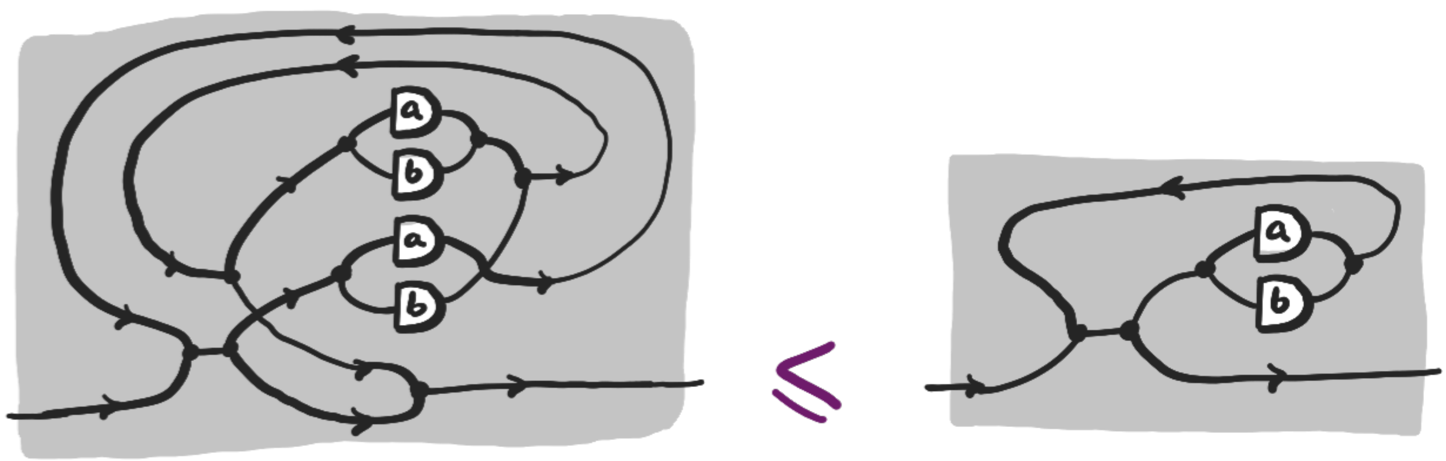
SOUNDNESS $\Uparrow$ $\Downarrow$ COMPLETENESS



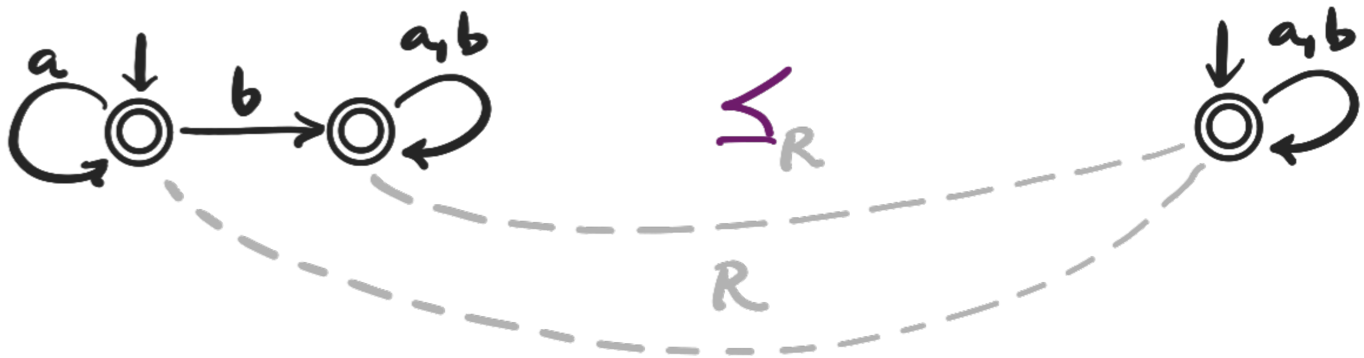
ONE-SLIDE SYNOPSIS



SOUNDNESS $\Uparrow$ $\Downarrow$ COMPLETENESS



ONE-SLIDE SYNOPSIS



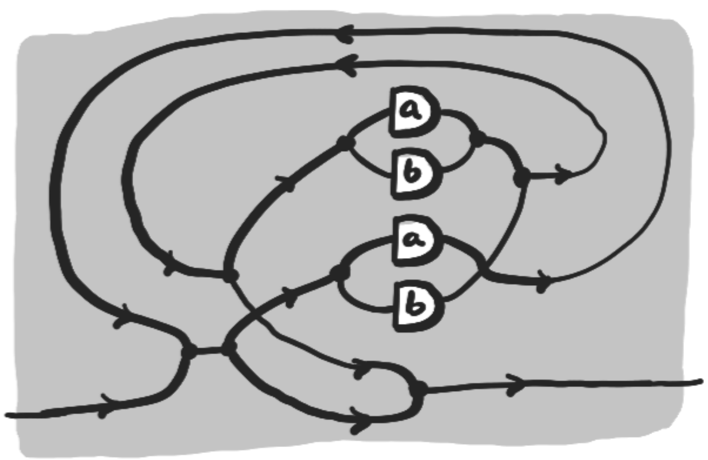
SOUNDNESS $\Uparrow$ $\Downarrow$ COMPLETENESS

$$a^*(b(a+b)^*+1) \leqslant (a+b)^*$$

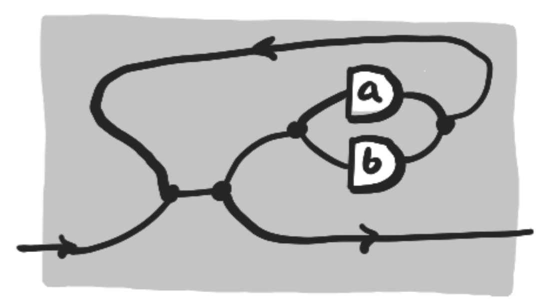
PREVIOUSLY...

recognised language

SOUNDNESS $\Uparrow$ $\Downarrow$ COMPLETENESS



$\leqslant$



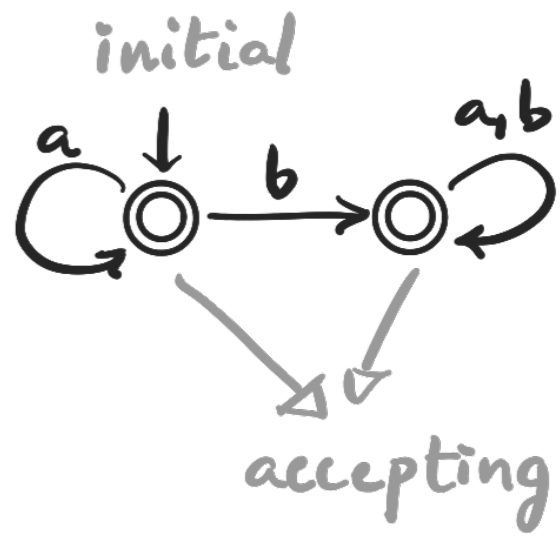
P. & Zanasi, A finite axiomatisation of finite-state automata using string diagrams, LMCS, 2023

NFA

$$(\Sigma, Q, \delta \subseteq Q \times \Sigma \times Q, q_0 \in Q, F \subseteq Q)$$

alphabet · finite set of states · transition relation · initial state · accepting states

E.g.



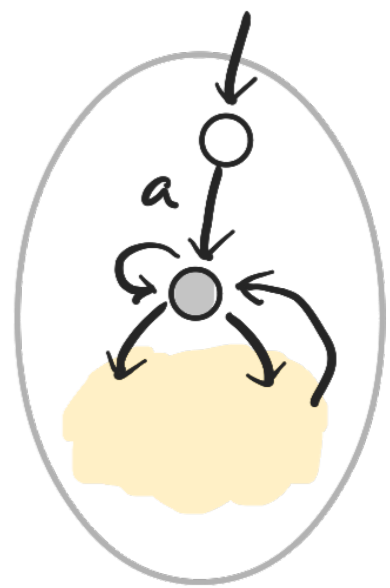
$$\Sigma = \{a, b\}$$

OPERATIONS ON NFA

- <u>Product</u>: $(q,s) \xrightarrow{a} (q',s')$ in $A \times B$ iff $q \xrightarrow{a} q'$ in A and $s \xrightarrow{a} s'$ in B.

- <u>Prefixing</u>:

$a \in \Sigma$

$a.$  $=$ 

SIMULATION

A <u>simulation</u> from A to B is a relation $R \subseteq Q^A \times Q^B$ s.t.

① if $(q,s) \in R$ and $q \in F^A$ then $s \in F^B$

② if $(q,s) \in R$ and $q \xrightarrow{a}_A q'$ then there exists $s' \in Q^B$ s.t. $s \xrightarrow{a}_B s'$ and $(q',s') \in R$.

③ $(q_0^A, q_0^B) \in R$

$\Rightarrow$ We write $A \leq_R B$ or $A \preceq B$

SIMILARITY

A and B are (two-way) <u>similar</u>, written $A \simeq B$, if

$$A \lesssim_R B \quad \text{and} \quad B \lesssim_S A$$

behaviours

<u>Theorem</u>. The set Ω of NFA modulo $\simeq$ form a <u>semi</u>-<u>lattice</u> with:

- $A \times B$ as meet
- 

as top

And prefixing is <u>monotone</u>.

2D SYNTAX FOR NFA

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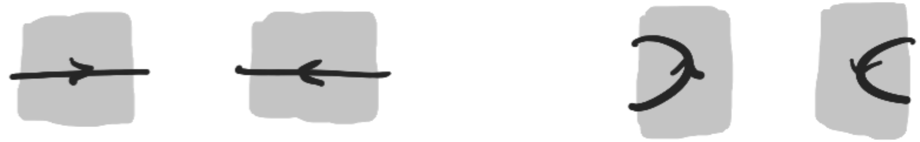
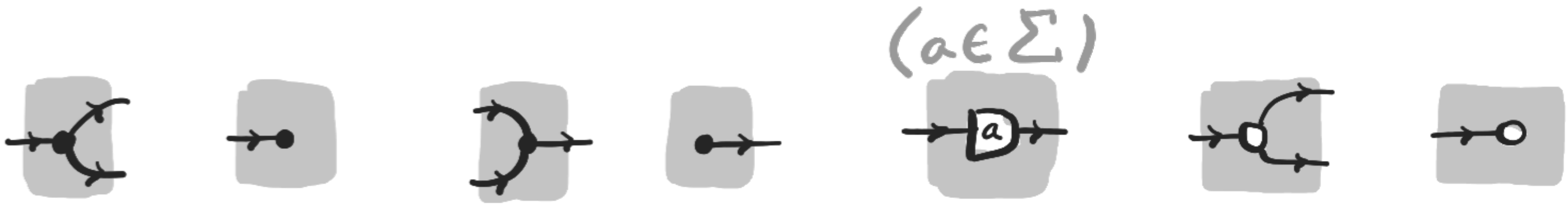
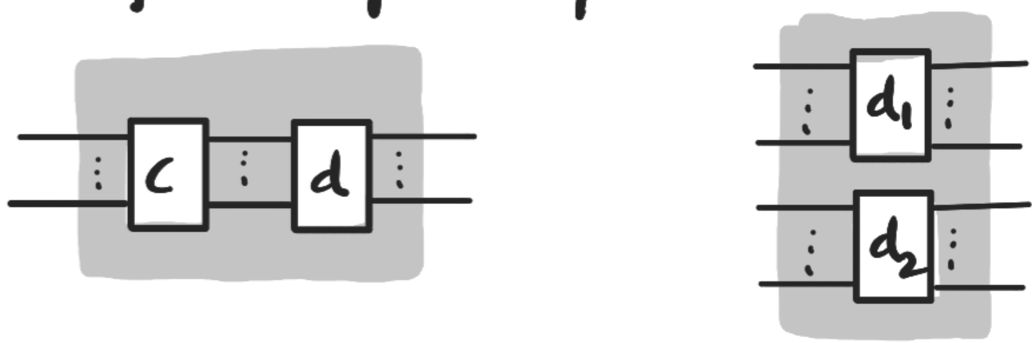




using two forms of composition:

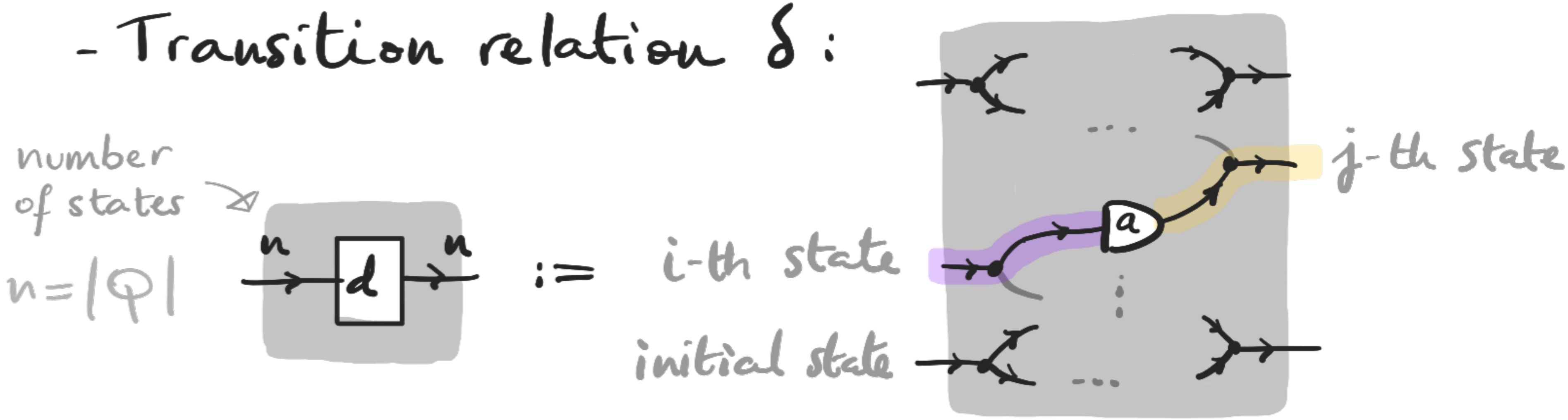


and wire crossings, e.g. 

P. & Zanasi, A finite axiomatisation of finite-state automata using string diagrams, LMCS, 2023

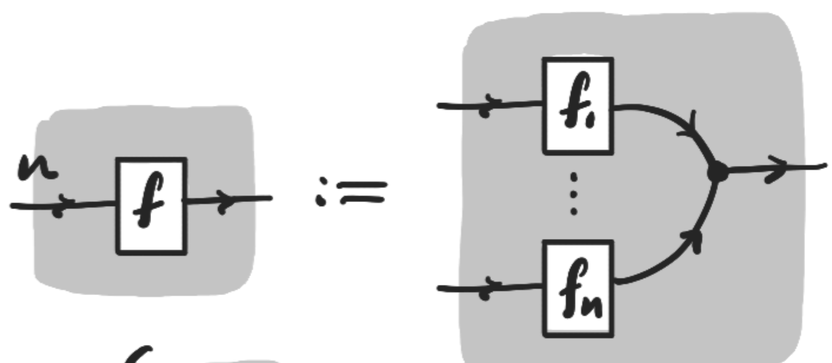
ENCODING NFA

- Transition relation δ:

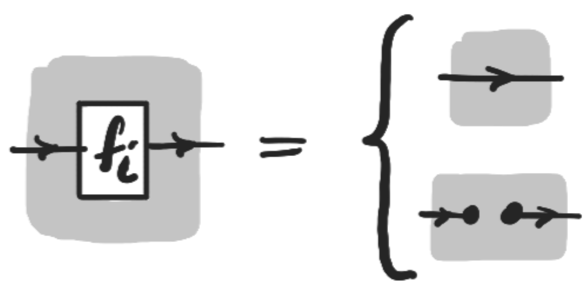


iff $(q_i, a, q_j) \in \delta$

- Final states:



where

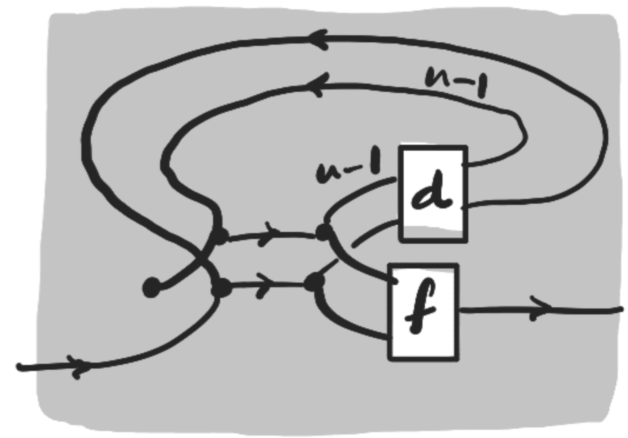


if $q_i \in F$,

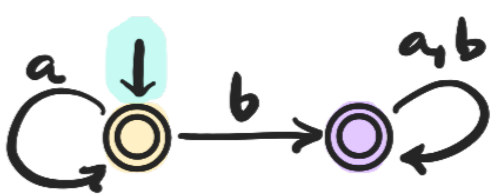
otherwise.

ENCODING NFA

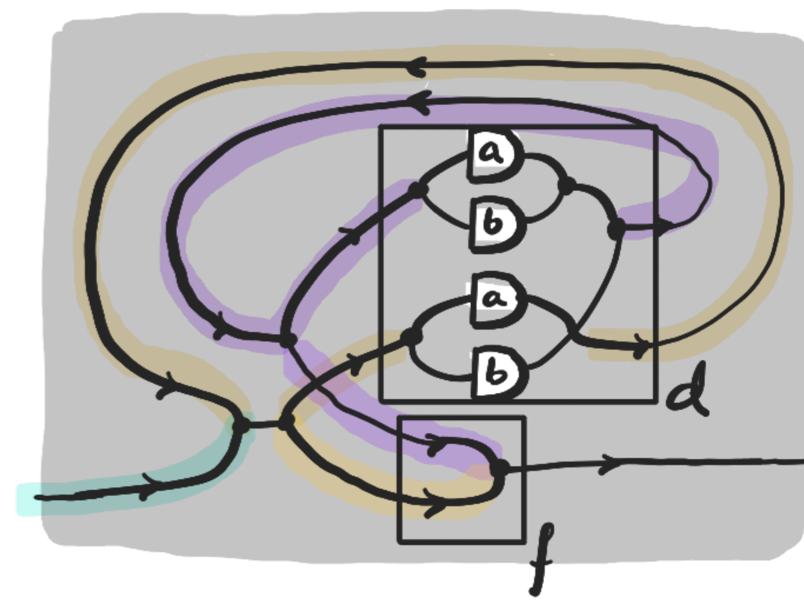
$(\Sigma, \{q_0, \dots, q_{n-1}\}, \delta, q_0, F) \mapsto$



For example:



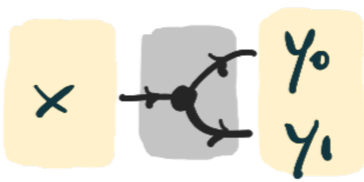
$\mapsto$



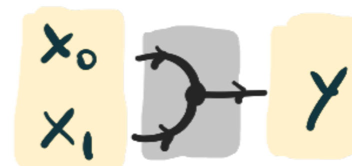
SEMANTICS

$x, y, x_i, y_i \in \Omega$

"Behaviours" $= \text{NFA up to} \simeq$



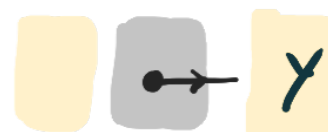
: x simulates y_0 and y_1



: x_0 and x_1 simulate y

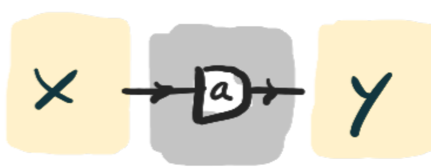
: no behaviour

($\perp :=$ )



: any behaviour

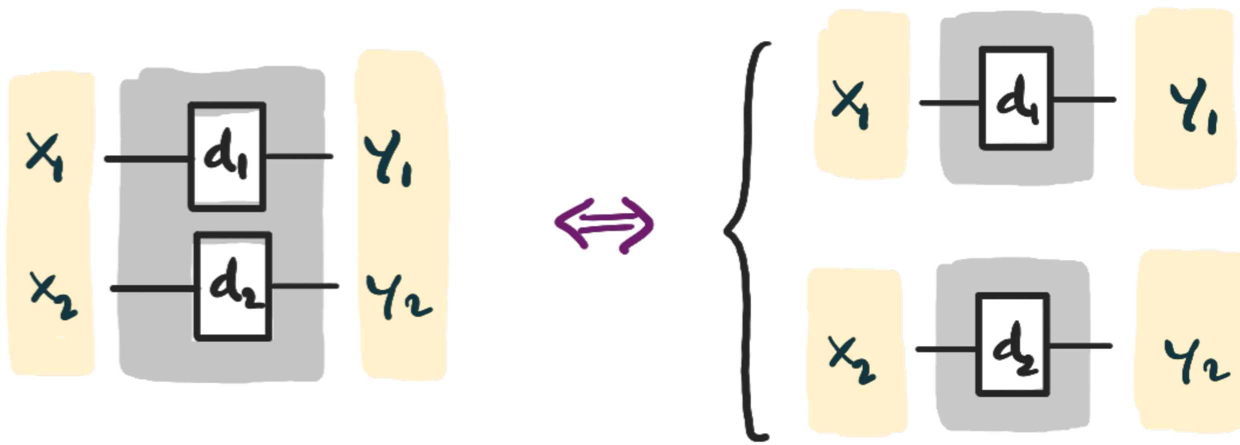
$(a \in \Sigma)$



: x can simulate $a.y$

"plumbing"
(cf. paper)

SEMANTICS

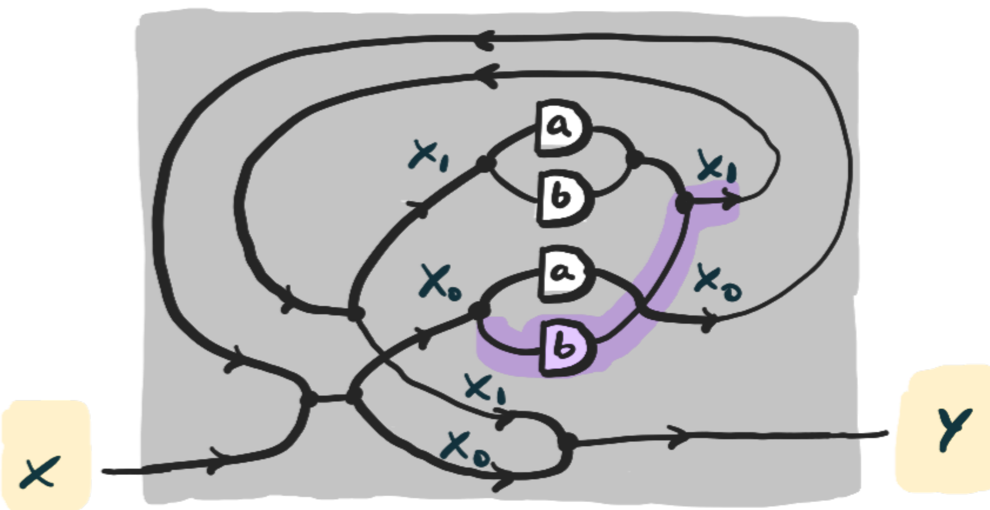


COMPOSITIONALITY

The behaviour of a composite diagram can be computed from the behaviour of its parts.

SEMANTICS

Diagram $\longmapsto$ Solution set of system of linear inequalities

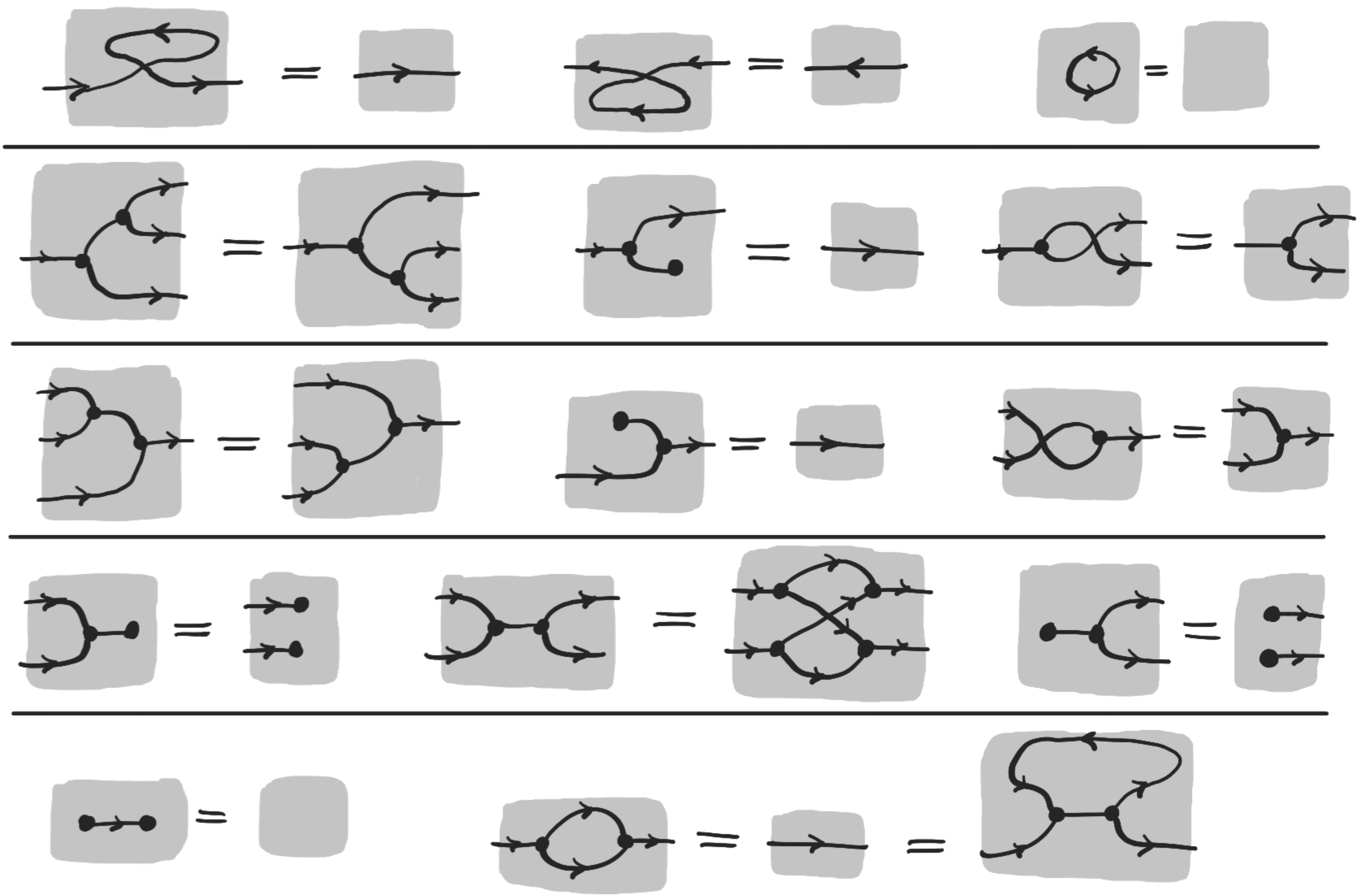


$\Longleftrightarrow$

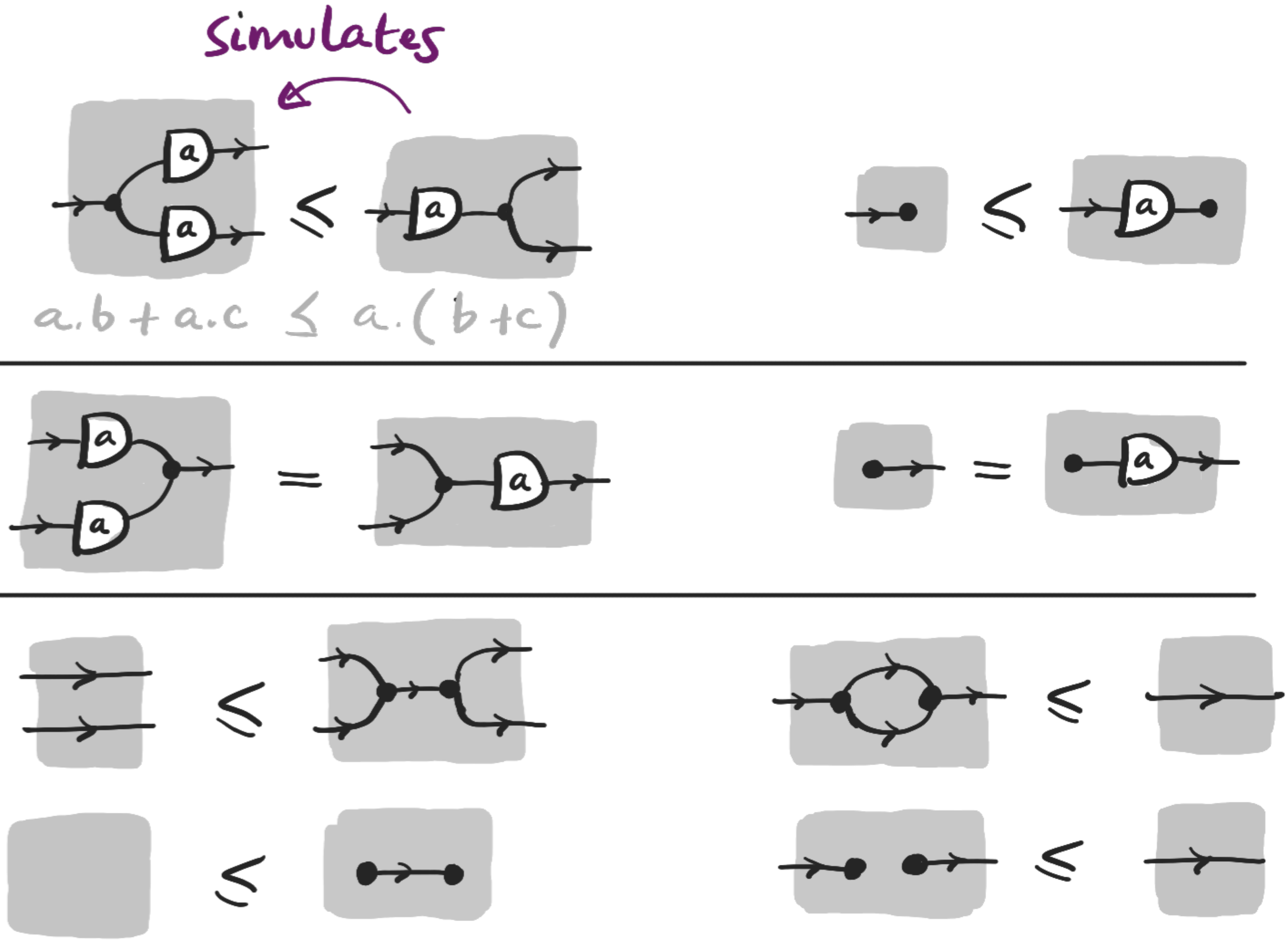
internal vars

$$\exists x_0, x_1 \left(x_0 \leq x, \; a \cdot x_0 \leq x_0, \; b \cdot x_1 \leq x_0, \; a \cdot x_1 \leq x_1, \; b \cdot x_1 \leq x_1, \; y \leq x_0, x_1 \right)$$

EQUATIONAL THEORY



EQUATIONAL THEORY



SOUNDNESS

Theorem. If $\boxed{c_A}$ and $\boxed{c_B}$ encode NFA A and B respectively, then

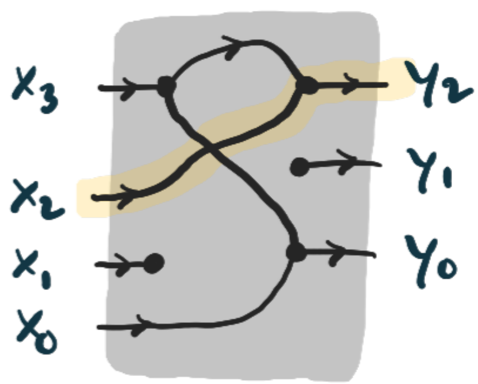
$$\boxed{c_A} \leqslant \boxed{c_B} \implies A \leq B$$

Proof. Using the semantics: we just need to check the validity of all axioms.

SIMULATIONS AS DIAGRAMS

A simulation is just a relation and relations correspond to diagrams in the fragment generated by

E.g.

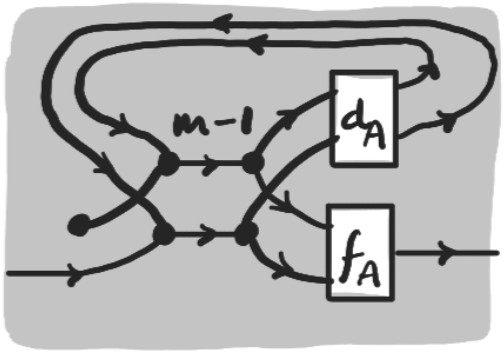


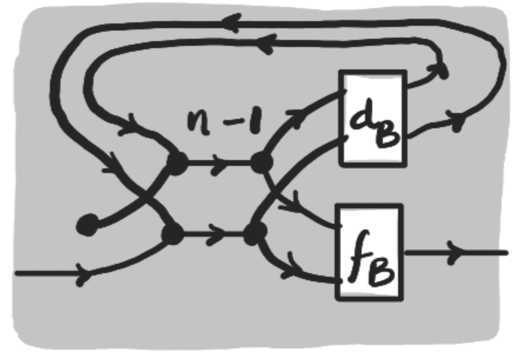
$\longleftrightarrow$

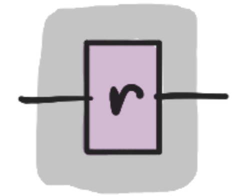
$$\{(x_3, y_2), (x_3, y_0), (x_2, y_2), (x_0, y_0)\}$$

(x_1, y_1 do not belong to the relation)

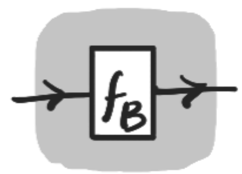
Simulations as diagrams

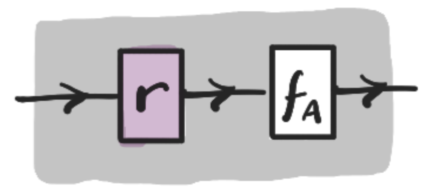
If 

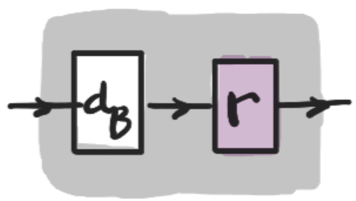
and 

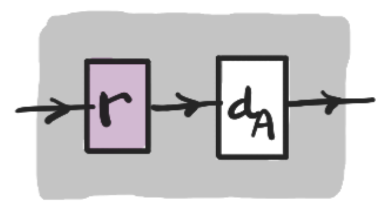
encode NFA A and B, and $A \leq_R B$, then there exists 

s.t.

① 

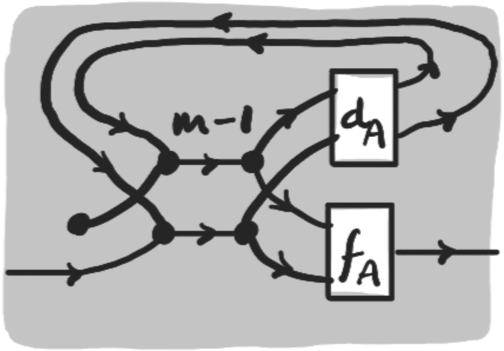
$\geqslant$ 

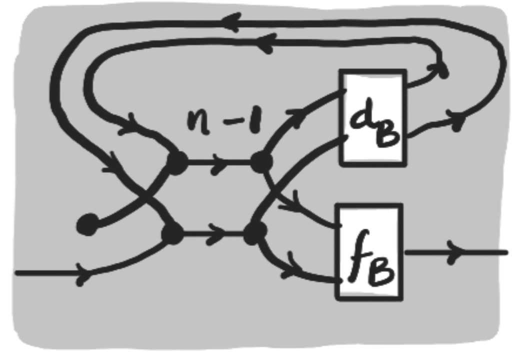
② 

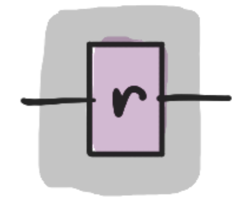
$\geqslant$ 

i.e. internalise properties of simulations.

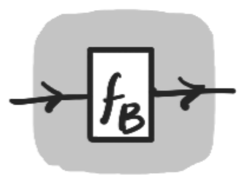
SIMULATIONS AS DIAGRAMS

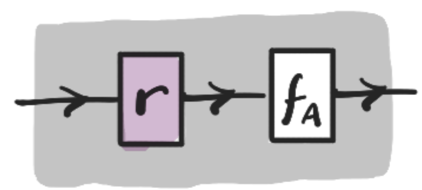
If 

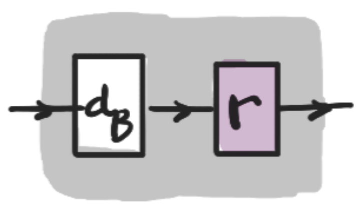
and 

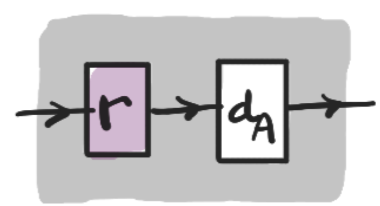
encode NFA A and B, and $A \leq_R B$, then there exists 

s.t.

1 

$\geqslant$ 

2 

$\geqslant$ 

BUT NOT ENOUGH...

2D SYNTAX FOR NFA

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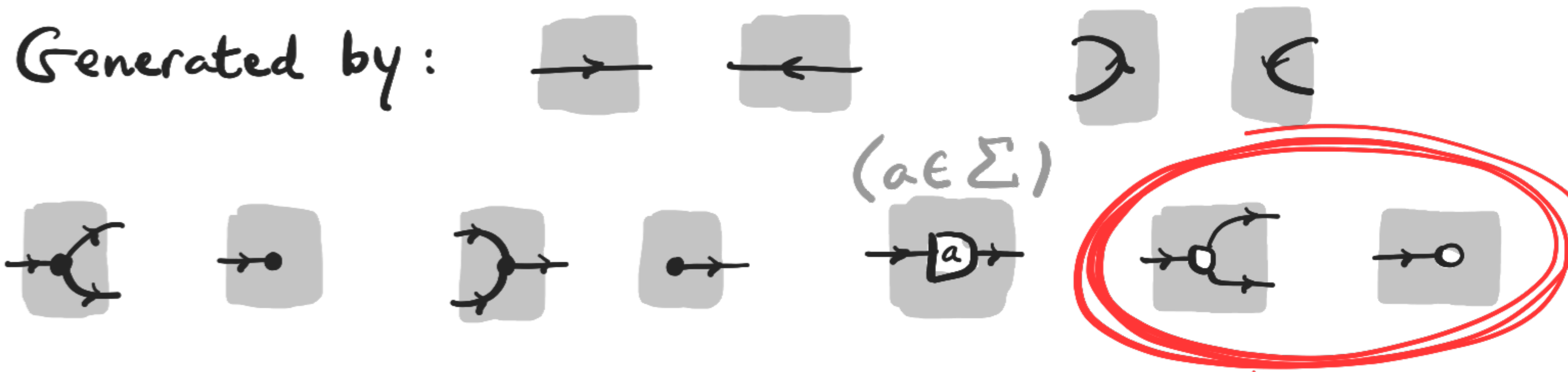
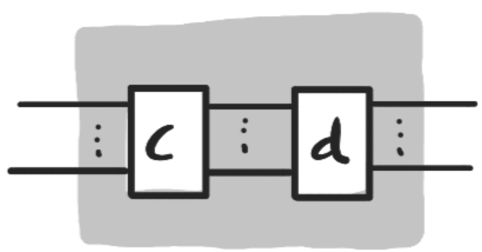
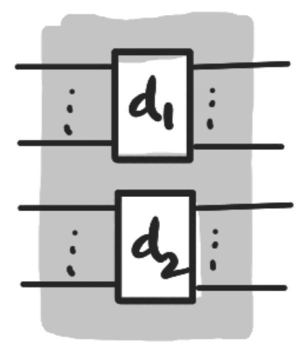


What about these?

using two forms of composition:





and wire crossings, e.g. 

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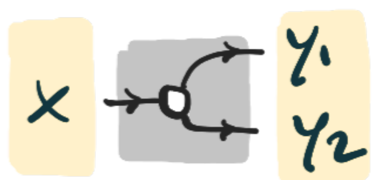
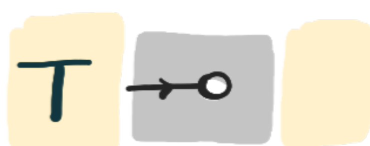
SEMANTICS, CONTINUED

product of NFA / meet of lattice



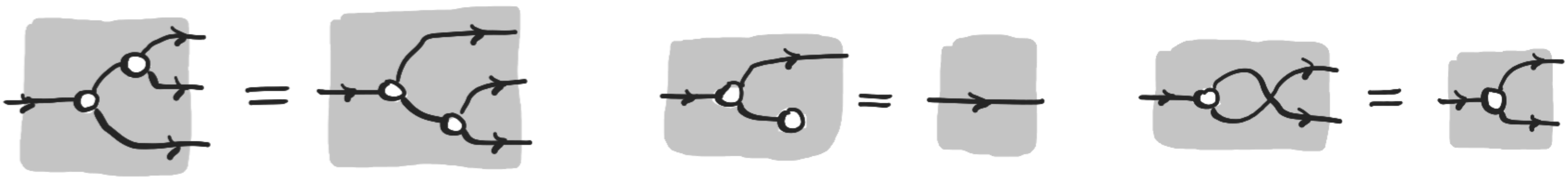
: x simulates $y_1 \times y_2$

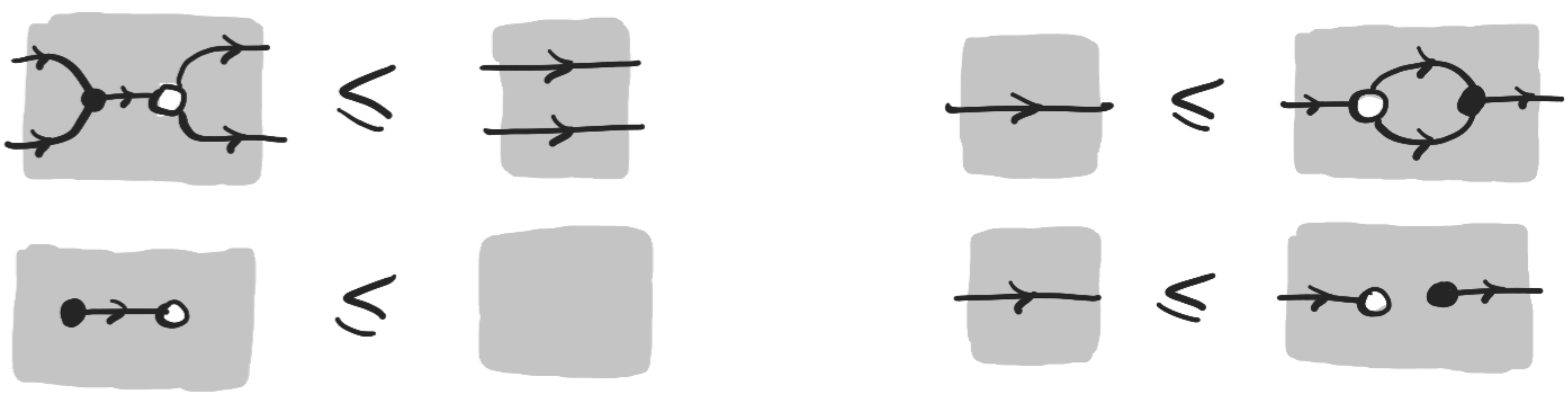


: behaves as

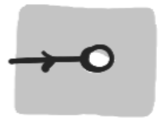
top of lattice

EQUATIONAL THEORY, CONTINUED





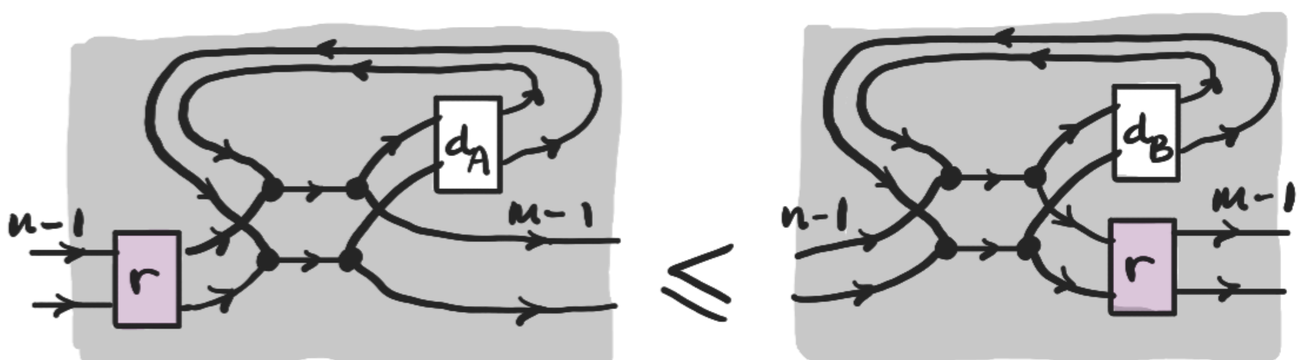
Simulations as diagrams, continued

Using   with the axioms above, we can show

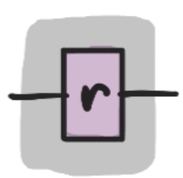
② 

one-step

$\Downarrow$ $\Downarrow$

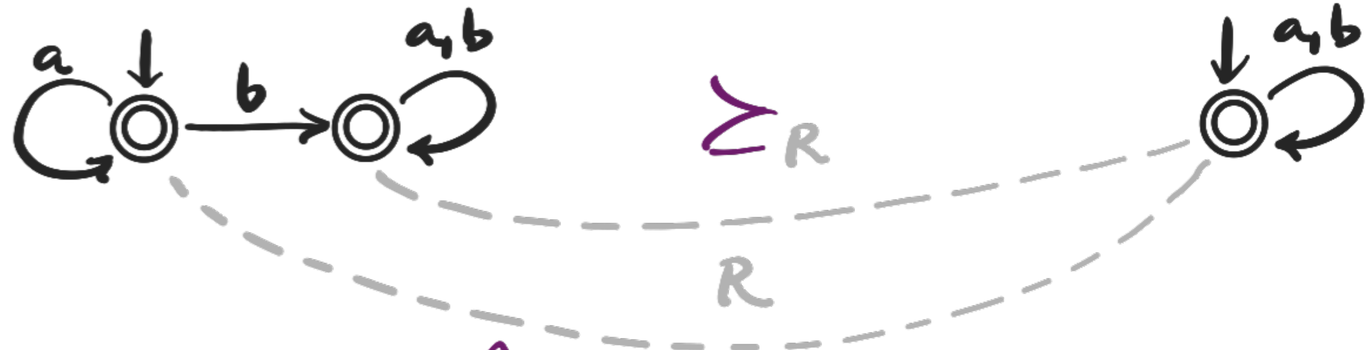


arbitrarily many steps

for 

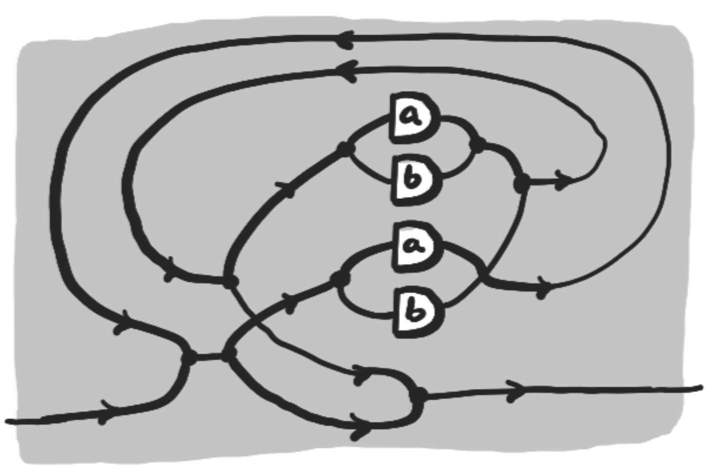
encoding a simulation $A \leq_R B$

WORKED EXAMPLE

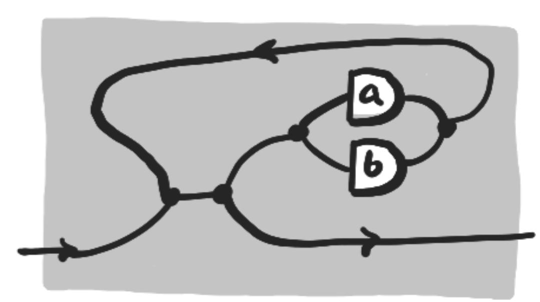


we have

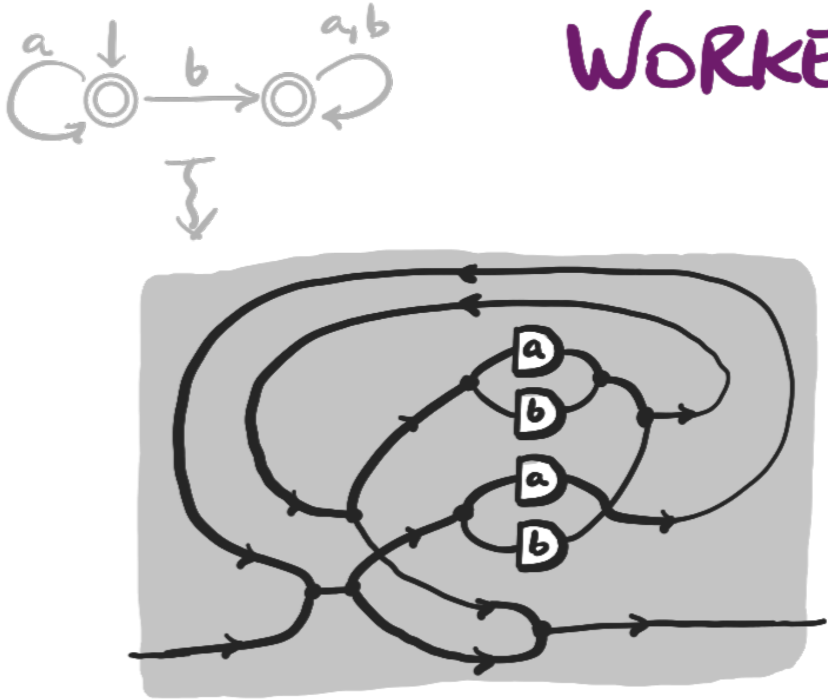
we want to show



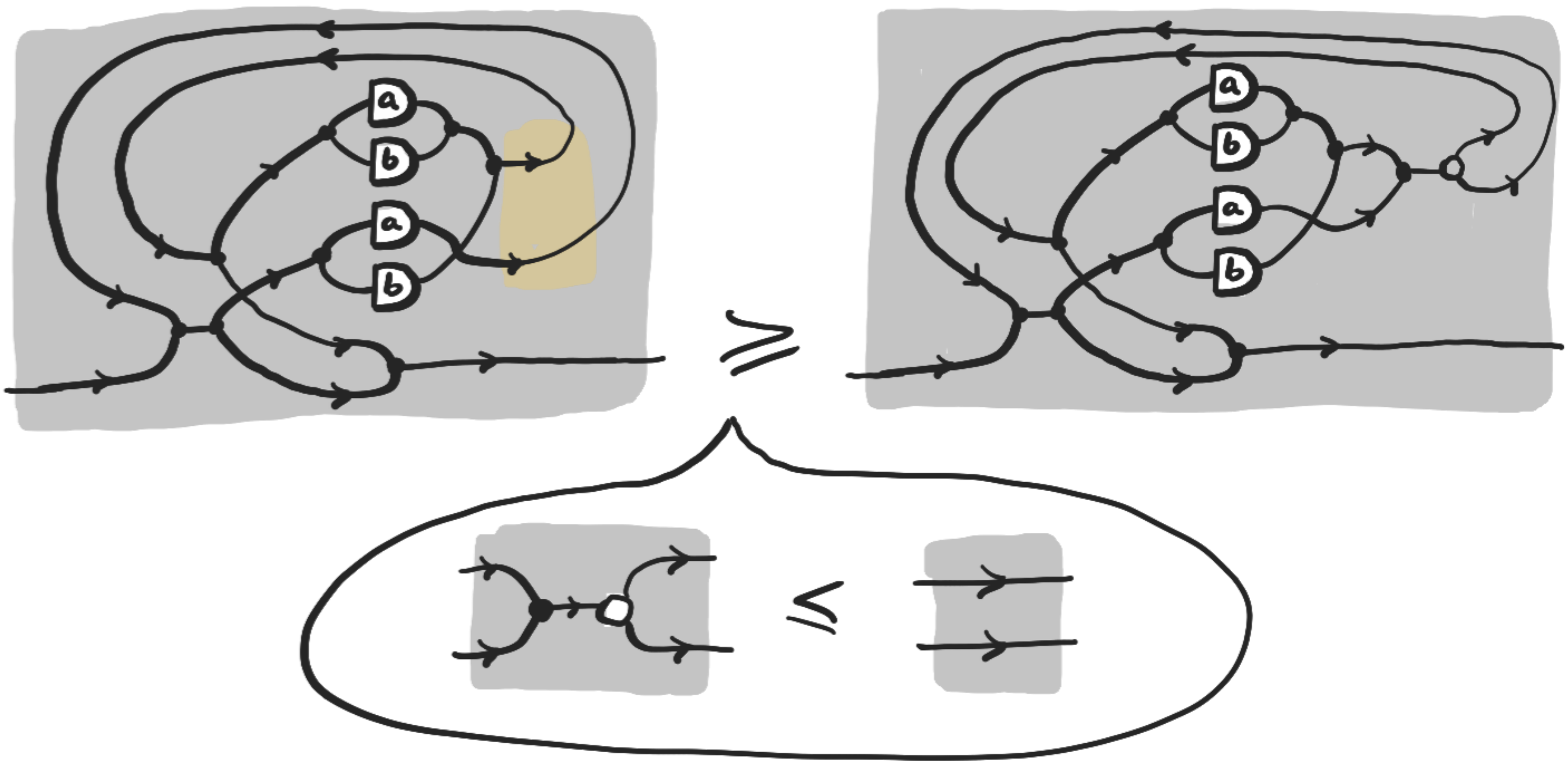
$\geq$



WORKED EXAMPLE



WORKED EXAMPLE



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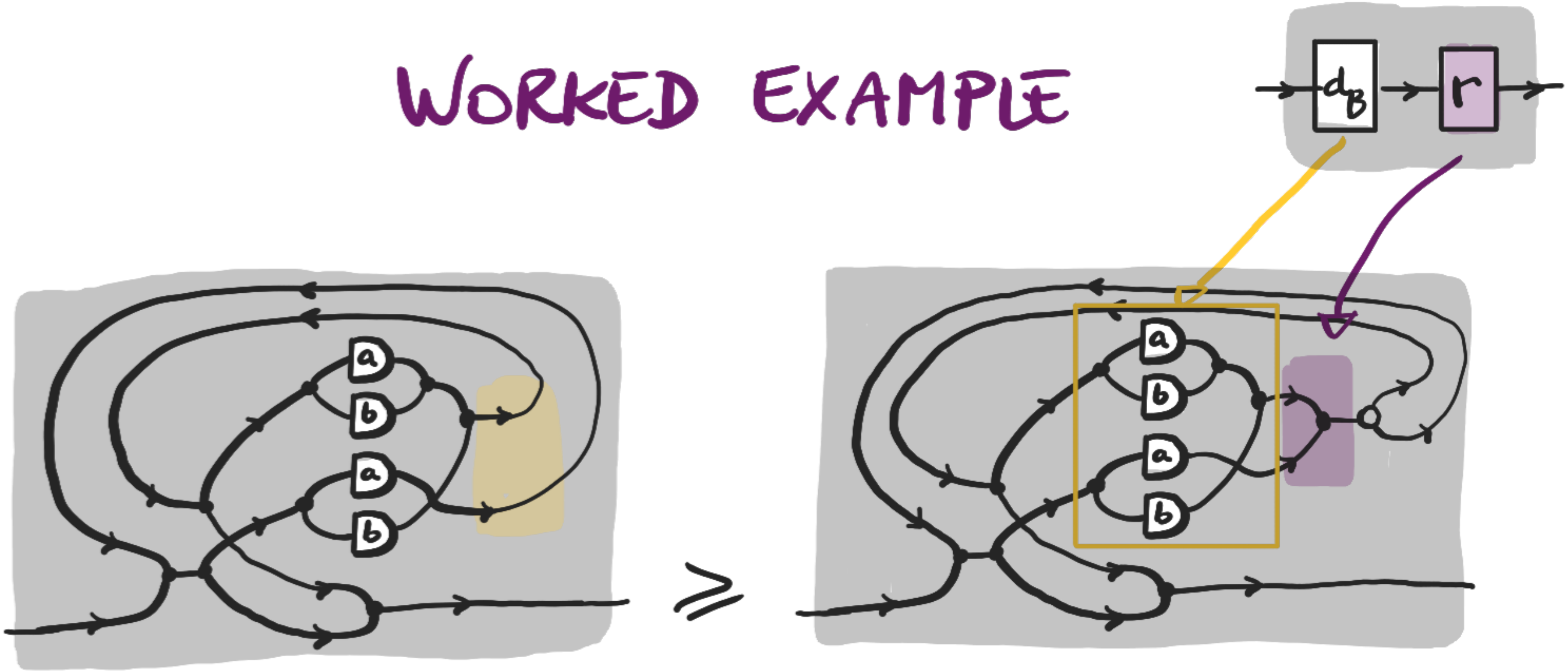
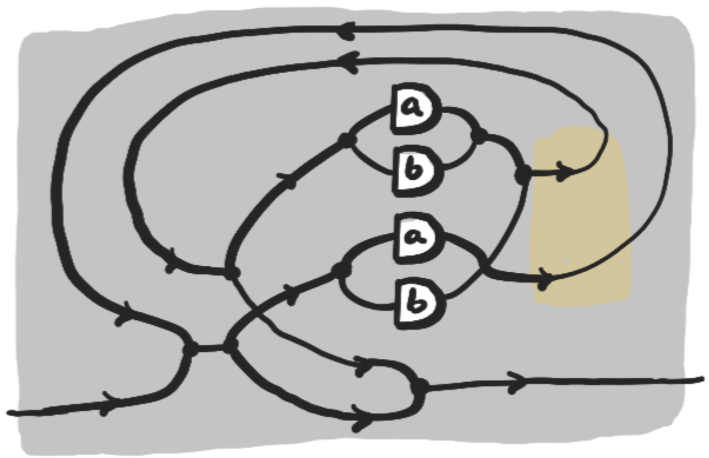
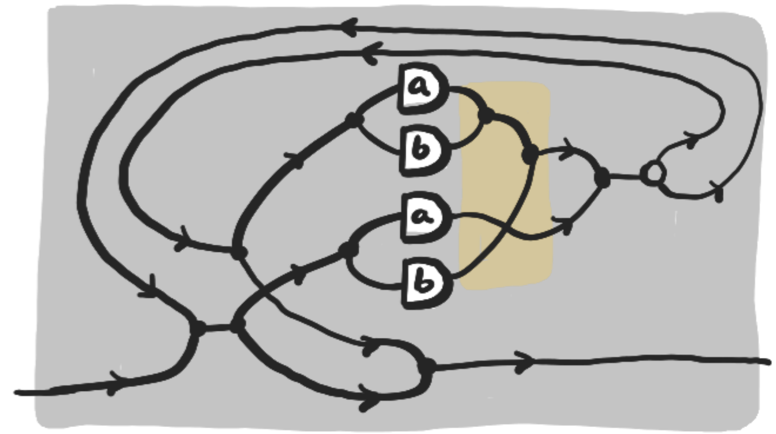


WORKED EXAMPLE



$\geqslant$



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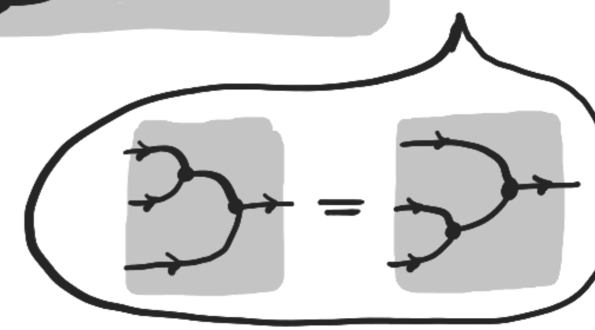

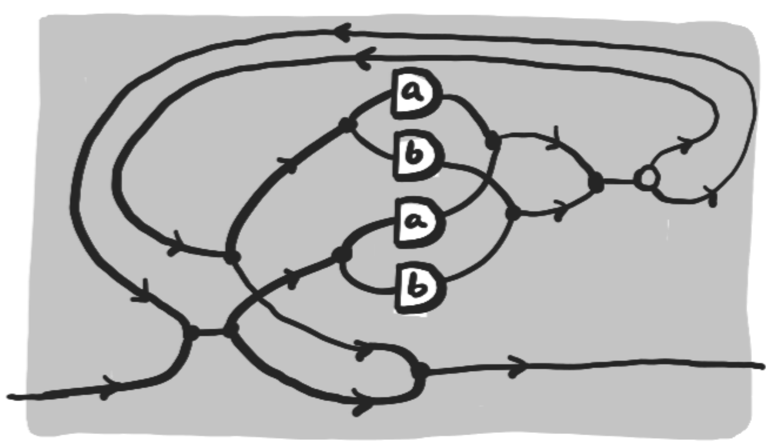

(N.B. we are showing

②

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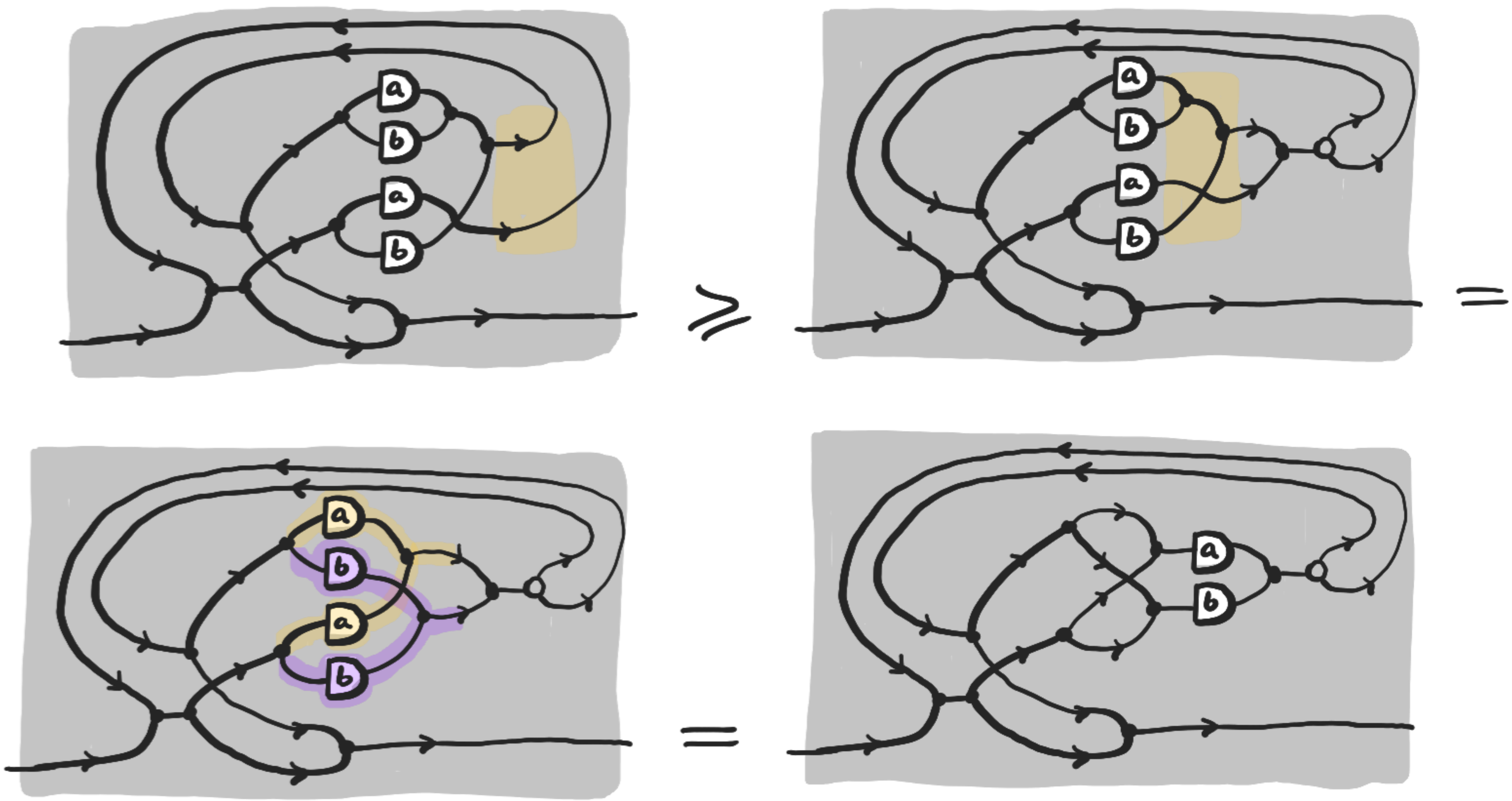
WORKED EXAMPLE



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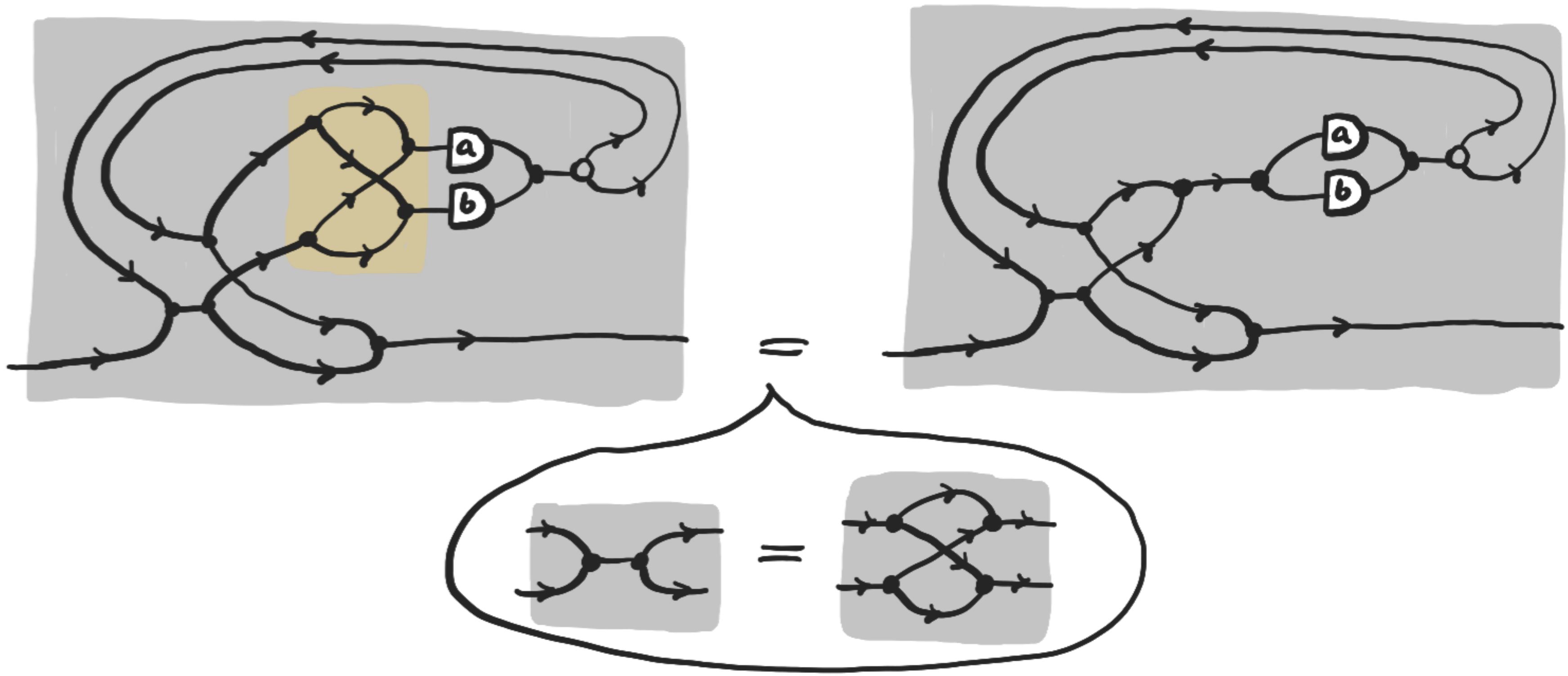

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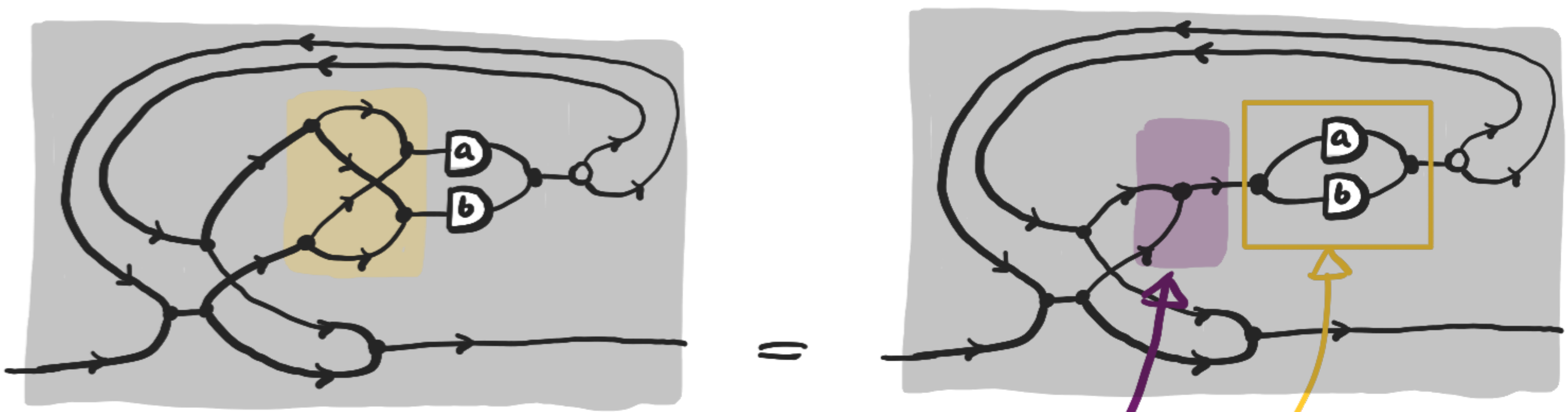
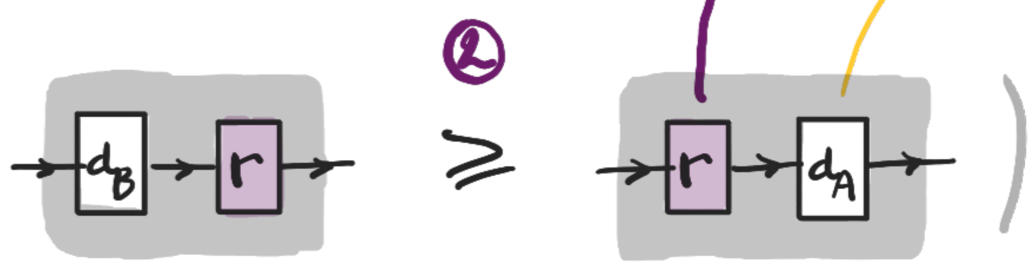


(we have shown



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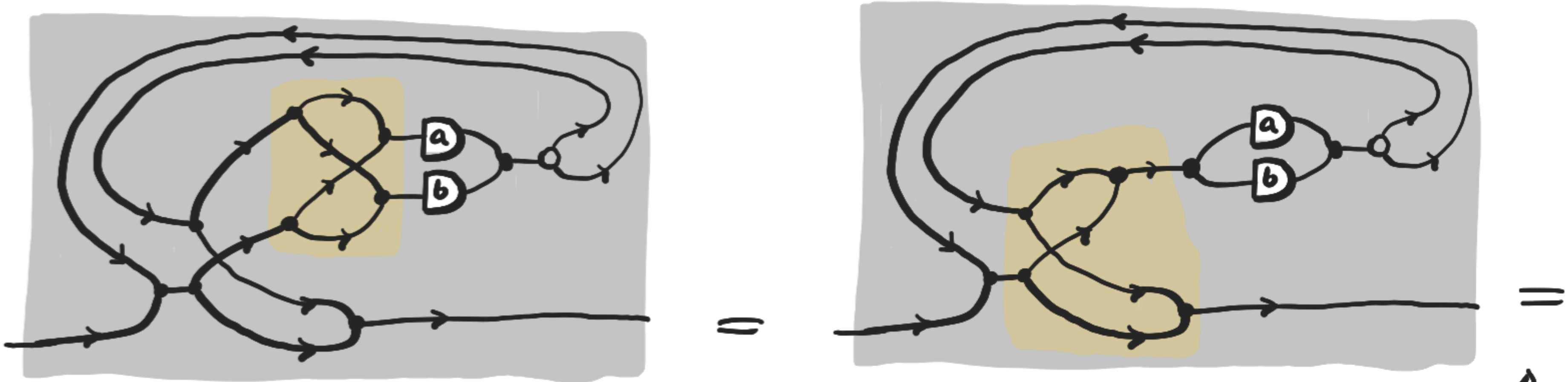

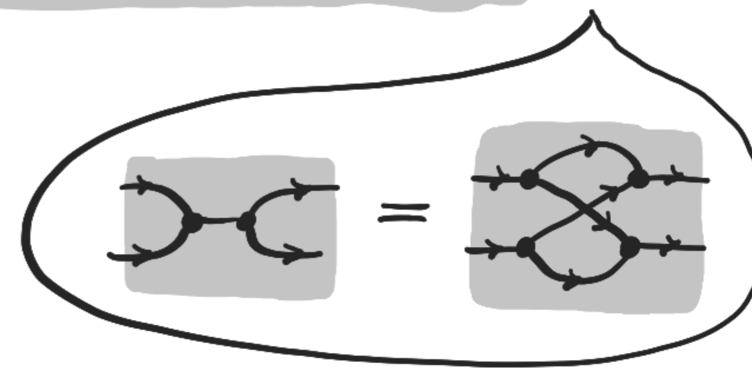

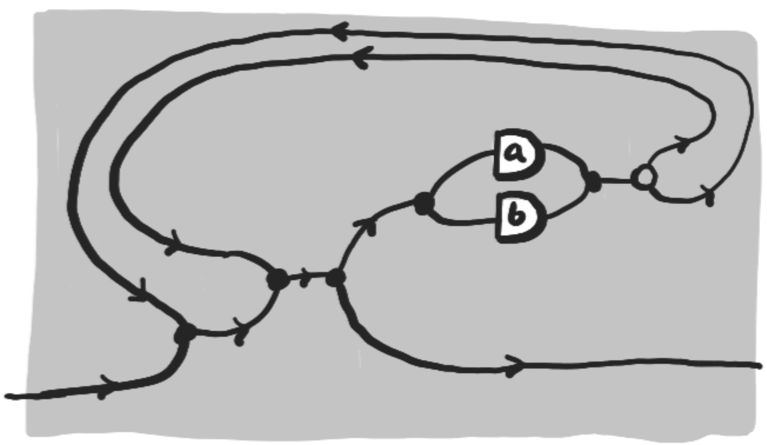

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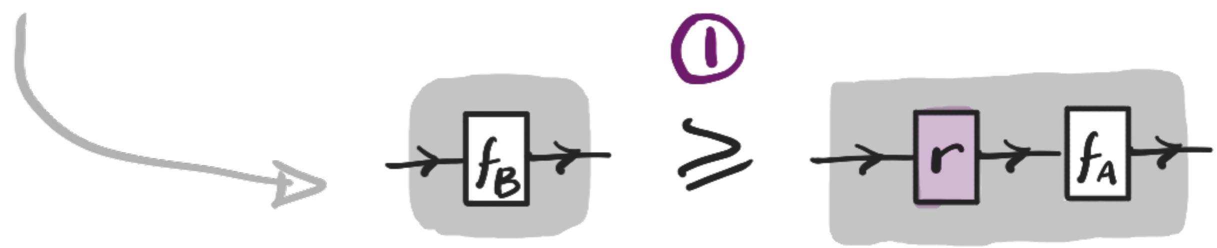

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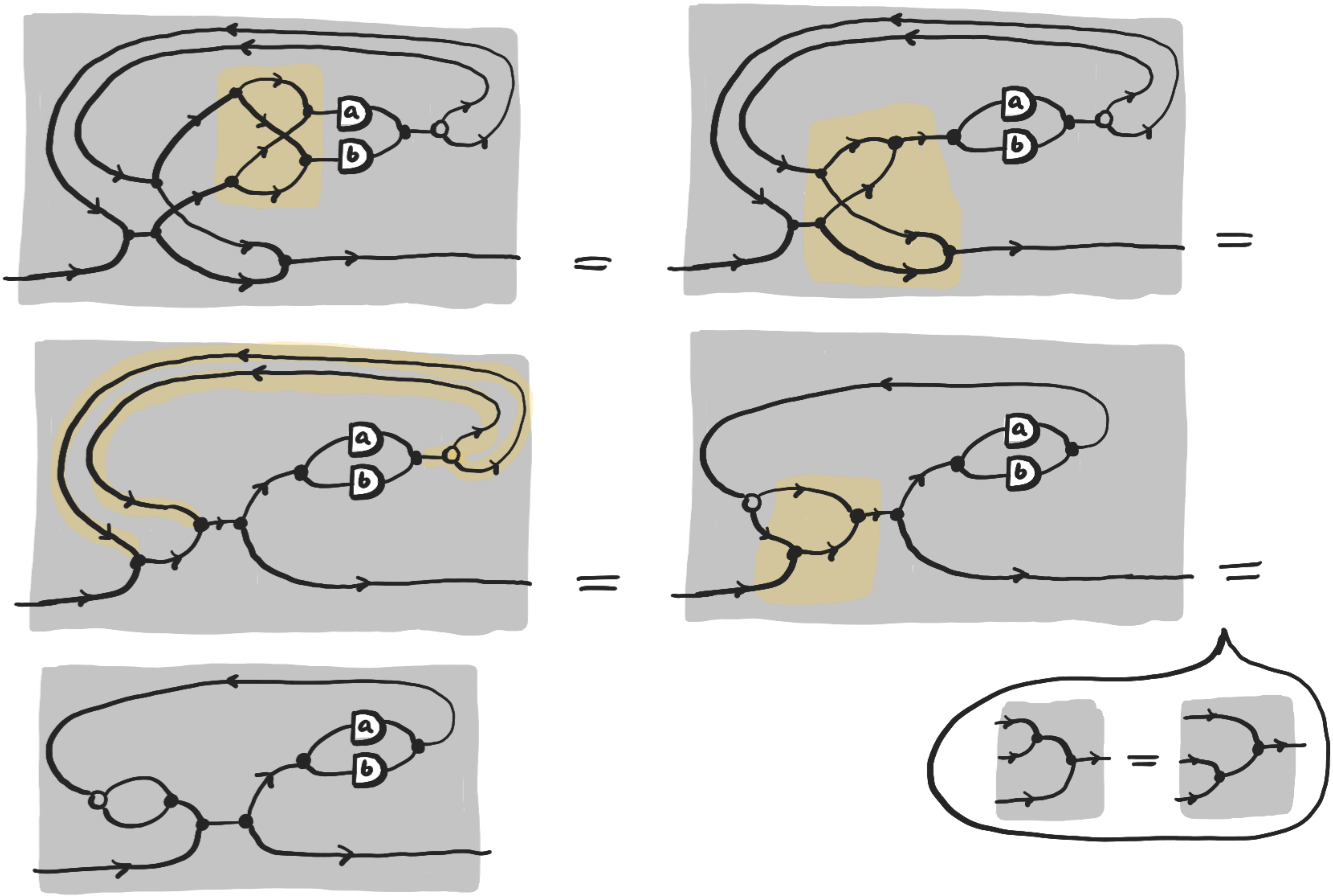

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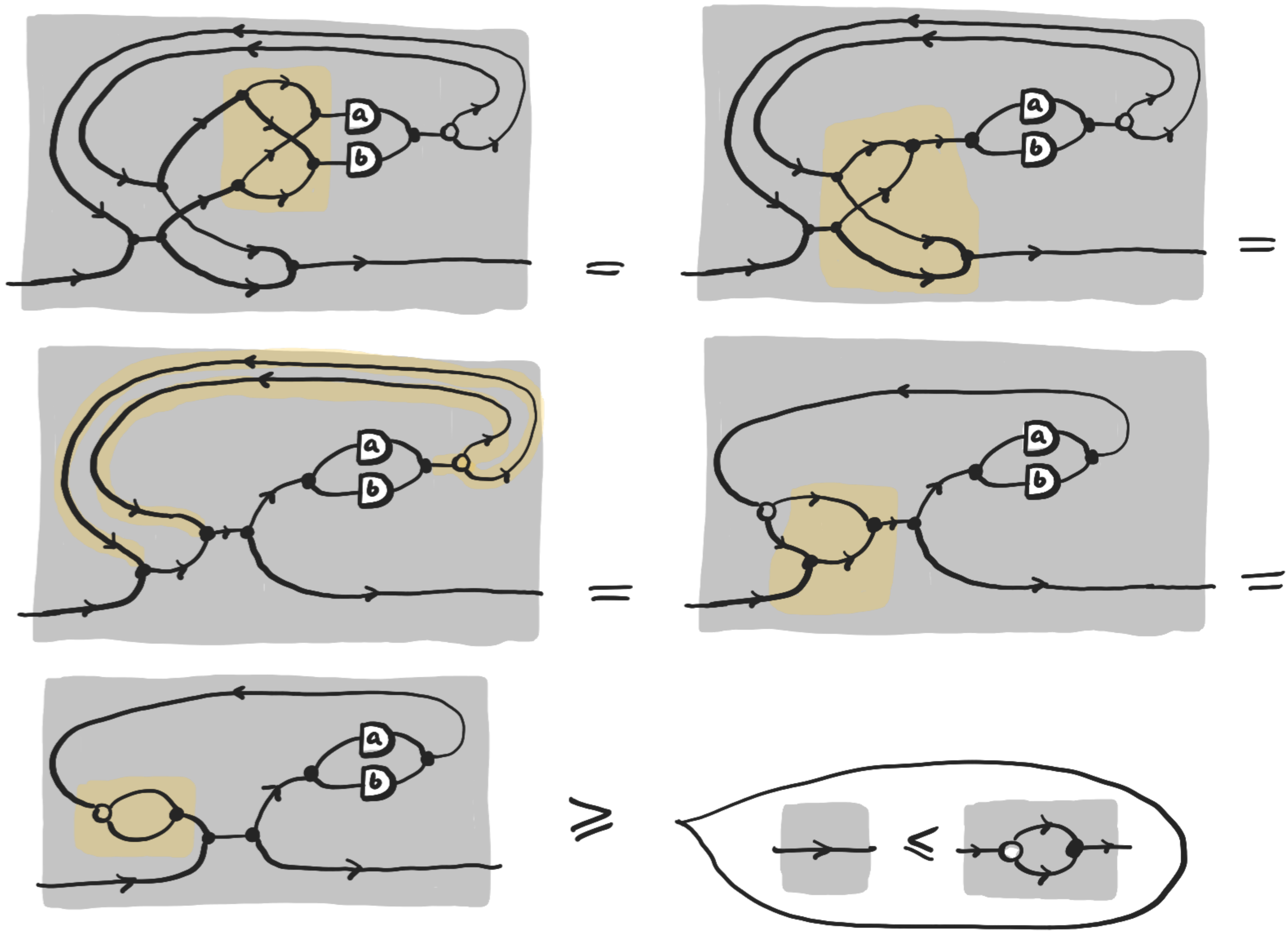

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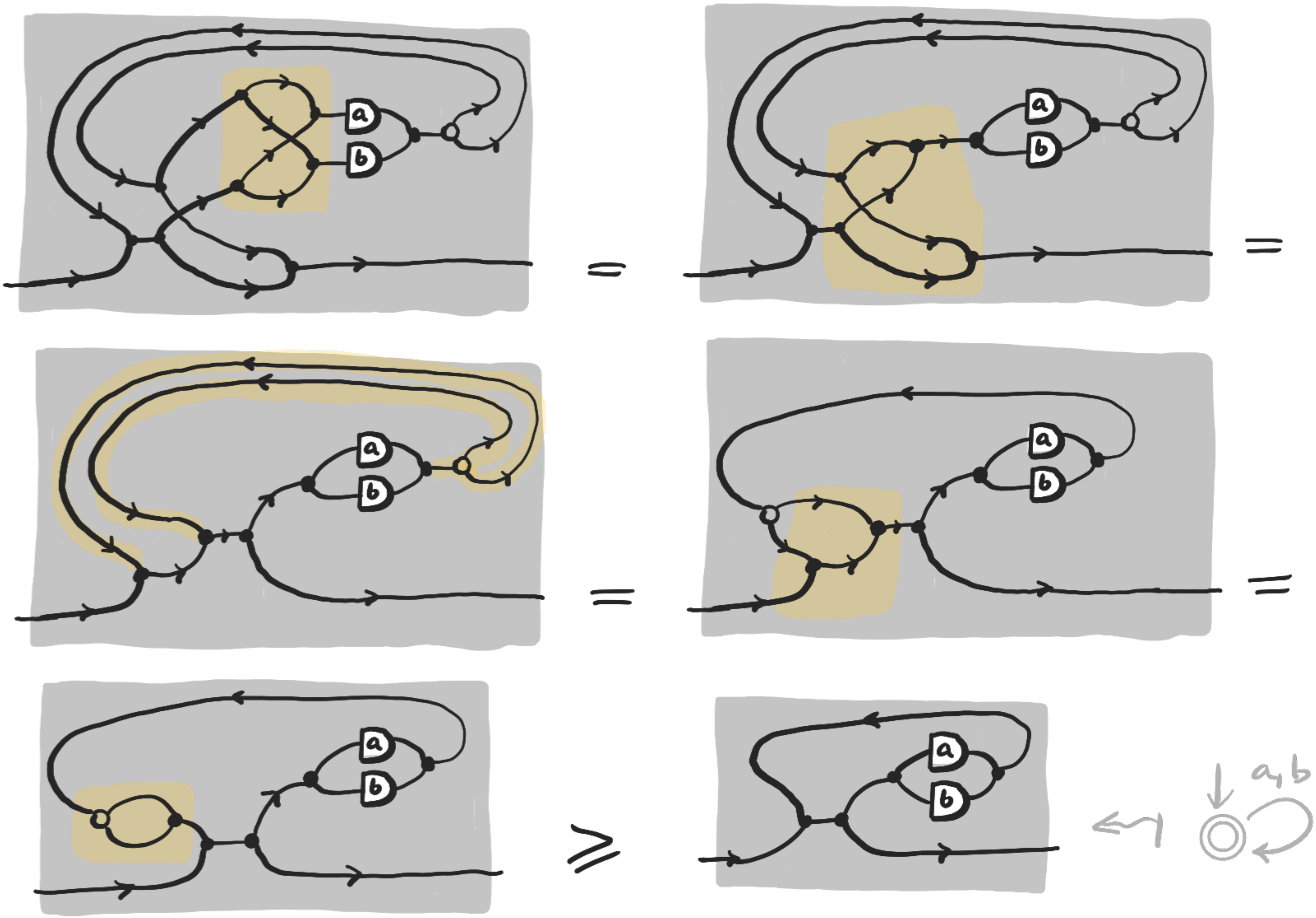


Completeness

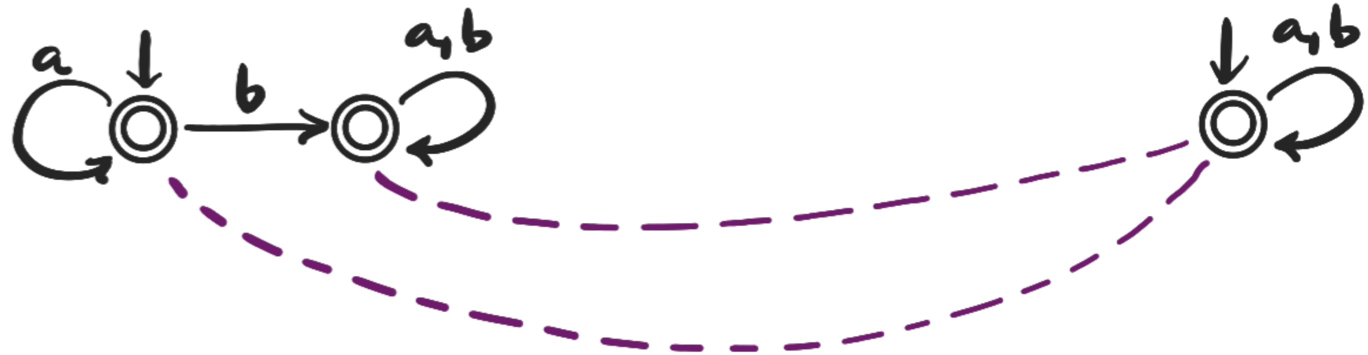
Theorem. If $-\boxed{c_A}-$ and $-\boxed{c_B}-$ encode NFA A and B respectively, then

$$A \leq B \implies -\boxed{c_A}- \leq -\boxed{c_B}-$$

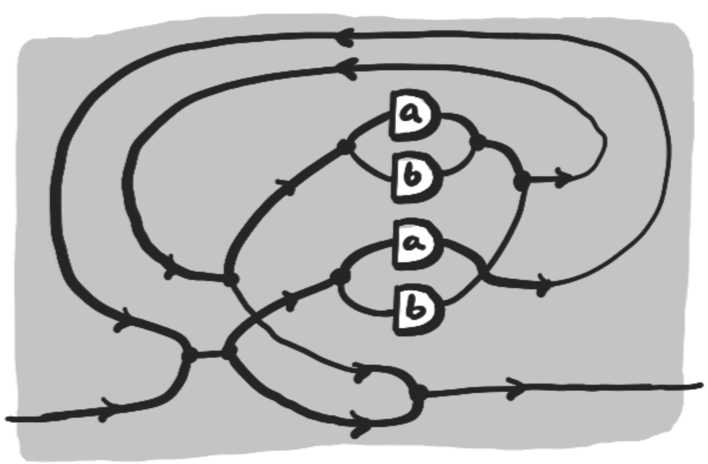
Proof idea: Via encoding simulations as diagrams and using the axioms above to show that they indeed behave as simulations syntactically.

FUTURE WORK

- Using an extended syntax to internalise proofs of simulation, equivalence, etc. in other models: KAT, GKAT, CKA, ...
- Bisimulation in a 2-categorical setting (i.e. with "proof-relevant inequalities" $\leq_R$)
- Formulate axioms as hypergraph-rewriting system (DPO).



THANK YOU! $\Downarrow$ $\Uparrow$ QUESTIONS ?



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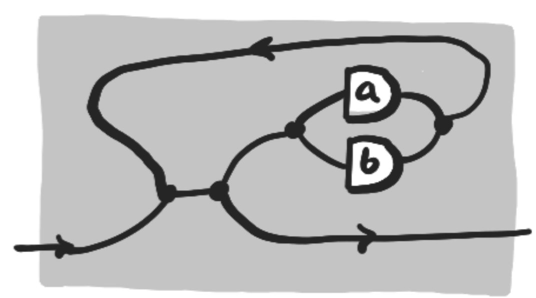

(1D) SYNTAX FOR NFA

- Regular expressions

$$e ::= e+e \mid e \cdot e \mid e^* \mid 0 \mid 1 \mid a \in \Sigma$$

- Milner's algebra of regular behaviours

fragment of CCS

$$e ::= e+e \mid \mu x.e \mid a.e \mid 0 \mid x$$

↳ has an axiomatisation

Frendrup & Jensen, A complete axiomatisation of simulation for regular CCS expressions, 2001